Buckling and Vibration of Unsymmetrically Laminated Cross-Ply Rectangular Plates

ROBERT M. JONES*
SMU Institute of Technology, Dallas, Texas

An exact solution and numerical results are presented for simply supported plates that are laminated unsymmetrically about their middle surface. The coupling between bending and extension induced by the lamination asymmetry substantially decreases buckling loads and vibration frequencies for common composite materials such as boron/epoxy and graphite/epoxy. For antisymmetric laminates, the effect of the coupling dies out rapidly as the number of layers is increased. For generally unsymmetric laminates, however, the effect of coupling dies out very slowly as the number of layers increases. That is, having a large number of layers is no guarantee that coupling will not seriously degrade buckling resistance and vibration frequencies. Thus, designers must include coupling between bending and extension in all analyses of unsymmetrically laminated plates.

Nomenclature†

a = plate length in the x-direction (Fig. 1)
A_{ij} = extensional stiffnesses of a laminated plate
b = plate length in the y-direction (Fig. 1)
B_{ij} = coupling stiffnesses of a laminated plate
D_{ij} = bending stiffnesses of a laminated plate
E_1 = Young's modulus in the 1-direction of a lamina
E_2 = Young's modulus in the 2-direction of a lamina
G_{12} = shearing modulus in the 1–2 plane of a lamina
m = number of buckle or vibration halfwaves in the x-direction
$\delta M_x, \delta M_{xy}, \delta M_y$ = variations in moments per unit length during buckling or vibration
n = number of buckle or vibration halfwaves in the y-direction
N = number of layers in a laminated plate
$\delta N_x, \delta N_{xy}, \delta N_y$ = variations in in-plane forces per unit length
$\bar{N}_x, \bar{N}_y$ = applied in-plane forces in the x- and y-directions, respectively
Q_{ij} = reduced stiffnesses of a lamina, Eq. (4)
$\delta u, \delta v, \delta w$ = variations of displacements in the x-, y-, and z-directions, respectively, during buckling or vibration
x, y, z = plate coordinates (Fig. 1)
ν_{12} = Poisson's ratio for contraction (expansion) in the 2-direction due to tension (compression) in the 1-direction
ν_{21} = Poisson's ratio for contraction (expansion) in the 1-direction due to tension (compression) in the 2-direction, Eq. (5)

Introduction

LAMINATED plates are an important structural element in both the aerospace and electronics industries. In aerospace applications where weight savings are of paramount importance, the advent of advanced fiber-reinforced composite materials such as boron/epoxy and graphite/epoxy has resulted in a dramatic increase in the use of laminated fiber-reinforced plates and other structural shapes. The composite materials are typically a combination of a usually light, weak, and flexible matrix material with a more dense, very strong, and stiff reinforcing material in fibrous or whisker form. Hence, high strength-to-weight and stiffness-to-weight ratios are readily obtained. In electronics applications, circuit boards typically have two or more different materials in laminated form. In both uses of laminated plates, lamination asymmetry can result in coupling between bending and extension of the laminate. This phenomenon is evidenced by bending of a laminate that is subjected to only in-plane forces or extension of a laminate that is bent by application of moments only. Another example is bending of a laminate that is subjected to a uniform temperature rise; this is the principle by which the common thermostat works.

The objective of this paper is to present a theory with some numerical results for buckling and vibration of laminated rectangular plates as depicted in Fig. 1. Attention is restricted to cross-ply laminates, i.e., laminates with laminae (plys) that have their principal material directions (one being the fiber direction) either at 0° or 90° to one of the sides of the plate as shown in Fig. 1. The term cross-ply refers to the over-all laminate orientation as well as to the relative juxtaposition of the adjacent laminae. Previous results[1–4] were limited to cross-ply laminates that are symmetric or antisymmetric about their

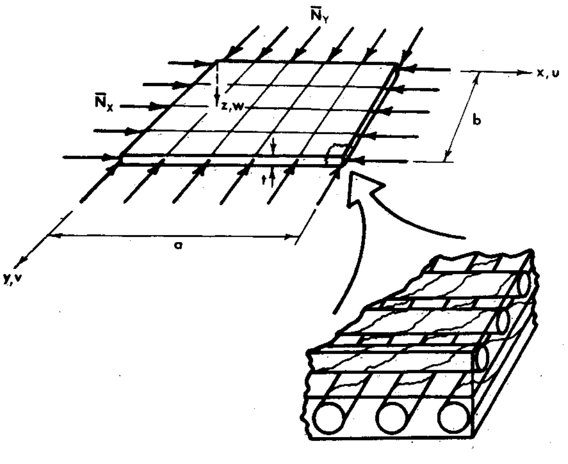

Fig. 1 Rectangular plate geometry and forces.

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Index categories: Structural Composite Materials (Including Coatings); Structural Stability Analysis; Structural Dynamic Analysis.

* Associate Professor of Solid Mechanics. Associate Fellow AIAA.

† A comma denotes partial differentiation with respect to the subscript following the comma. The prefix δ signifies the variation during buckling or vibration of the symbol which follows.

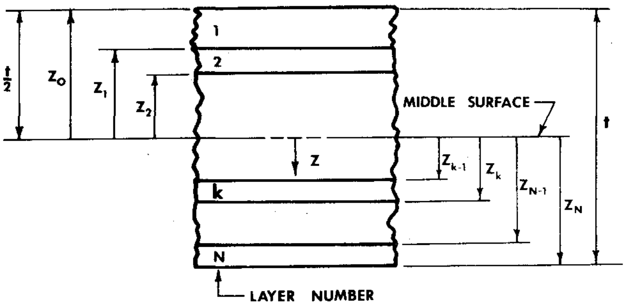

Fig. 2 Geometry of an N-layered laminate.

middle surface as defined in Fig. 2 and in addition have equal thickness layers. This restriction is removed in the present theory in which any sequence of 0° and 90° layers with any thickness can be considered. Accordingly, the description unsymmetrically laminated cross-ply rectangular plate is appropriate.

Further description of the differences between the various classifications of cross-ply laminates is appropriate because of the similarity of the phrases used to designate each class. First of all, the adjective "regular" is suggested by the writer to describe laminates with equal thickness laminae as in the top half of Fig. 3. Irregular or nonregular laminates have laminae of arbitrary, different thicknesses. Hereafter, dropping the adjective "regular" implies that the laminae are of unequal thickness, i.e., the term "irregular" will not be explicitly used, but is the default adjective. Also shown in Fig. 3 are examples of symmetric, antisymmetric, and unsymmetric laminates. Symmetric laminates have both geometric and material property symmetry about the middle surface defined in Fig. 2. Antisymmetric laminates have geometric symmetry and material property antisymmetry. Unsymmetric laminates could have geometric symmetry, but do not have material property symmetry or antisymmetry.

In this paper, the equations governing buckling and vibration of unsymmetrically laminated cross-ply plates are solved for S2 simply supported edge boundary conditions. General results are presented for buckling and vibration of regular antisymmetric cross-ply plates. Whitney[1-4] presented general results for vibration of regular antisymmetric cross-ply plates, but the results in the various references listed are somewhat inconsistent. The present vibration results are included for the sake of completeness and to correct some minor inaccuracies in Whitney's results. His conclusion is indeed correct that the degrading influence of coupling between bending and extension rapidly dies out as the number of laminae in the laminate is increased. His numerical results are qualitatively correct, but are slightly inaccurate.

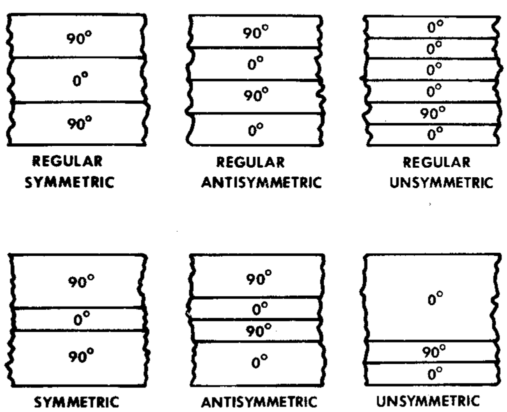

Fig. 3 Contrast between symmetric, antisymmetric, and unsymmetric cross-ply laminates.

Results for an example unsymmetrically laminated cross-ply plate are also presented herein and contrasted with the apparent viewpoint of many people that coupling between bending and extension rapidly vanishes as the number of layers increases and, in fact, is virtually of no consequence for many practical unsymmetric laminates.

Derivation of Buckling and Vibration Criterion

The differential equations governing buckling and vibrations of laminated plates are well known. They are commonly expressed in terms of variations of in-plane forces and moments which can be subsequently expressed in terms of variations of displacements during buckling or vibration. An important step in the alternative (and more useful) form of the equations is the use of the familiar laminated plate stiffnesses. The solution to the governing differential equations for unsymmetrically laminated cross-ply plates with S2 simply supported edge boundary conditions is simple and straightforward. The result is a closed form buckling and vibration criterion that is applicable for various combinations of in-plane forces.

Governing Differential Equations

The governing differential equations for buckling and vibration of plates subjected to in-plane forces $\bar{N}_x$ and $\bar{N}_y$ when prebuckling curvatures are ignored are[4]

$$\left.\begin{aligned} \delta N_{x,x}+\delta N_{xy,y}&=0\\ \delta N_{xy,x}+\delta N_{y,y}&=0\\ \delta M_{x,xx}+2\delta M_{xy,xy}+\delta M_{y,yy}-\bar{N}_x\delta w_{,xx}-\bar{N}_y\delta w_{,yy}&=\rho\delta w_{,tt}\end{aligned}\right\}\quad(1)$$

where the acceleration term is, of course, ignored for buckling problems; $\bar{N}_x$ and $\bar{N}_y$ (positive in compression) are ignored for unstressed free vibration problems; and all terms are considered for free vibration problems in which the equilibrium state has prestress terms $\bar{N}_x$ and $\bar{N}_y$.

The variations in forces and moments during buckling or vibration for an unsymmetrical cross-ply laminate are

$$\begin{Bmatrix}\delta N_x\\ \delta N_y\\ \delta N_{xy}\\ \delta M_x\\ \delta M_y\\ \delta M_{xy}\end{Bmatrix}=\begin{bmatrix}A_{11}&A_{12}&0&B_{11}&B_{12}&0\\ A_{12}&A_{22}&0&B_{12}&B_{22}&0\\ 0&0&A_{66}&0&0&B_{66}\\ B_{11}&B_{12}&0&D_{11}&D_{12}&0\\ B_{12}&B_{22}&0&D_{12}&D_{22}&0\\ 0&0&B_{66}&0&0&D_{66}\end{bmatrix}\begin{Bmatrix}\delta\varepsilon_x\\ \delta\varepsilon_y\\ \delta\gamma_{xy}\\ \delta\kappa_x\\ \delta\kappa_y\\ \delta\kappa_{xy}\end{Bmatrix}\quad(2)$$

(antisymmetric cross-ply laminates have $A_{11}=A_{22}$, $B_{12}=0$, $B_{22}=-B_{11}$, $B_{66}=0$ and $D_{11}=D_{22}$) in which the plate extensional, coupling, and bending stiffnesses are

$$(A_{ij},B_{ij},D_{ij})=\int_{-t/2}^{t/2}Q_{ij}(1,z,z^2)\,dz\quad(3)$$

The reduced stiffnesses, Q_{ij}, of an individual lamina are expressed in terms of the lamina principal material properties as

$$\left.\begin{aligned} Q_{11}&=E_1/(1-\nu_{12}\nu_{21})\\ Q_{12}&=\nu_{12}E_2/(1-\nu_{12}\nu_{21})\\ Q_{22}&=E_2/(1-\nu_{12}\nu_{21})\\ Q_{66}&=G_{12}\end{aligned}\right\}\quad(4)$$

where

$$\nu_{21}=\nu_{12}E_2/E_1\quad(5)$$

so there are only four independent material properties for an orthotropic lamina. Note that the 1- and 2-directions coincide with either the x- or y-directions of a cross-ply laminate depending on the orientation of the fibers (for angle-ply laminates, the principal material directions of each lamina are oriented at some angle to the x-direction; hence, the Q_{ij} are more complex).

The variations in in-plane strains and changes of curvatures during buckling or vibration are

$$\delta\varepsilon_x=\delta u_{,x}\qquad\delta\varepsilon_y=\delta v_{,y}\qquad\delta\gamma_{xy}=\delta u_{,y}+\delta v_{,x}\quad(6)$$

$$\delta\kappa_x=-\delta w_{,xx}\qquad\delta\kappa_y=-\delta w_{,yy}\qquad\delta\kappa_{xy}=-2\delta w_{,xy}\quad(7)$$

Accordingly, the governing differential equations in terms of variations in displacements during buckling or vibration are

$$\left.\begin{aligned}
&A_{11}\delta u_{,xx}+A_{12}\delta v_{,yx}-B_{11}\delta w_{,xxx}-B_{12}\delta w_{,yyx}+\\
&\qquad A_{66}(\delta u_{,yy}+\delta v_{,xy})-2B_{66}\delta w_{,xyy}=0\\
&A_{66}(\delta u_{,yx}+\delta v_{,xx})-2B_{66}\delta w_{,xyx}+A_{12}\delta u_{,xy}+A_{22}\delta v_{,yy}-\\
&\qquad B_{12}\delta w_{,xxy}-B_{22}\delta w_{,yyy}=0\\
&B_{11}\delta u_{,xxx}+B_{12}\delta v_{,yxx}-D_{11}\delta w_{,xxxx}-D_{12}\delta w_{,yyxx}+\\
&\quad 2B_{66}(\delta u_{,yxy}+\delta v_{,xxy})-4D_{66}\delta w_{,xyxy}+B_{12}\delta u_{,xyy}+\\
&\quad B_{22}\delta v_{,yyy}-D_{12}\delta w_{,xxyy}-D_{22}\delta w_{,yyyy}-\\
&\qquad \bar{N}_x\delta w_{,xx}-\bar{N}_y\delta w_{,yy}=\rho\delta w_{,tt}
\end{aligned}\right\} \quad (8)$$

Solution of Governing Differential Equations

The free vibration of an elastic continuum is harmonic in time so all variations in middle surface displacements are of the form

$$\left.\begin{aligned}
\delta u(x,y,t)&=\delta u(x,y)\cos(\omega t+\phi)\\
\delta v(x,y,t)&=\delta v(x,y)\cos(\omega t+\phi)\\
\delta w(x,y,t)&=\delta w(x,y)\cos(\omega t+\phi)
\end{aligned}\right\} \quad (9)$$

Note that the harmonic function in time can be represented in many ways, e.g., $\cos\omega t$, $\sin\omega t$, $e^{i\omega t}$, $A\cos\omega t+B\sin\omega t$, etc. The end result on the differential equation that must be solved in spatial (x, y) coordinates is identical in all cases.

The $S2$ simply supported edge boundary conditions are[5]

$$\delta w=0 \qquad \delta M_n=0 \qquad \delta N_n=0 \qquad \delta u_t=0 \quad (10)$$

where the subscripts n and t denote normal and transverse to the edge, respectively. These boundary conditions and the governing differential equations, Eq. (8), are satisfied by the following variations in displacements

$$\left.\begin{aligned}
\delta u(x,y)&=\bar{u}\cos(m\pi x/a)\sin(n\pi y/b)\\
\delta v(x,y)&=\bar{v}\sin(m\pi x/a)\cos(n\pi y/b)\\
\delta w(x,y)&=\bar{w}\sin(m\pi x/a)\sin(n\pi y/b)
\end{aligned}\right\} \quad (11)$$

(as can readily but tediously be verified by substitution) if

$$\rho\omega^2+\bar{N}_x\left(\frac{m\pi}{a}\right)^2+\bar{N}_y\left(\frac{n\pi}{b}\right)^2=T_{33}+\frac{2T_{12}T_{13}T_{23}-T_{11}T_{23}{}^2-T_{22}T_{13}{}^2}{T_{11}T_{22}-T_{12}{}^2} \quad (12)$$

in which

$$\left.\begin{aligned}
T_{11}&=A_{11}(m\pi/a)^2+A_{66}(n\pi/b)^2\\
T_{12}&=(A_{12}+A_{66})(m\pi/a)(n\pi/b)\\
T_{13}&=-(B_{12}+2B_{66})(m\pi/a)(n\pi/b)^2-B_{11}(m\pi/a)^3\\
T_{22}&=A_{22}(n\pi/b)^2+A_{66}(m\pi/a)^2\\
T_{23}&=-(B_{12}+2B_{66})(m\pi/a)^2(n\pi/b)-B_{22}(n\pi/b)^3\\
T_{33}&=D_{11}(m\pi/a)^4+2(D_{12}+2D_{66})(m\pi/a)^2(n\pi/b)^2+D_{22}(n\pi/b)^4
\end{aligned}\right\} \quad (13)$$

The buckling and vibration criterion in Eq. (12) is the exact nontrivial solution to the homogeneous equations in T_{ij} and the loading parameters that results from the aforementioned substitutions.

For buckling problems, the buckling loads result when $\omega=0$ in Eq. (12). As an example, for $\bar{N}_x$ only

$$\bar{N}_x=[a/(m\pi)]^2\text{RHS} \quad (14)$$

where RHS is the right-hand side of Eq. (12). Or for combinations of $\bar{N}_x$ and $\bar{N}_y$

$$\bar{N}_x=\text{RHS}/[(m\pi/a)^2+k(n\pi/b)^2] \quad (15)$$

wherein $\bar{N}_y=k\bar{N}_x$ (k is a specified constant). In all cases, the lowest buckling load must be found by minimizing the right-hand side of an equation like Eqs. (14) or (15) for integer values of m and n which determine the buckle mode shape.[6] Because of the numerous parameters involved and the many repetitive calculations in the minimization procedure, a computer program (BURP for Buckling and vibration of Unsymmetrically laminated Rectangular Plates) was written; numerical results are discussed in the next section.

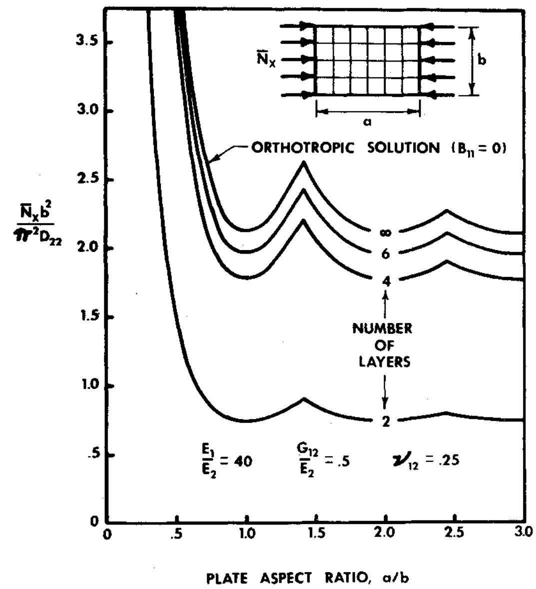

Fig. 4 Uniaxial buckling loads of rectangular antisymmetric cross-ply laminated plates.

For free vibration problems without prestress, the vibration frequencies are

$$\omega=(\text{RHS}/\rho)^{1/2} \quad (16)$$

where the associated mode shapes are the variations in displacements, Eq. (11), with the mode numbers m and n substituted corresponding to the values used in RHS to determine ω. The fundamental natural frequency is the lowest ω and is obtained by the minimization procedure involving integer values of m and n (Ref. 6). The solution, Eq. (12), reduces to Whitney's solution for antisymmetric cross-ply laminates[1] when the proper laminate stiffnesses are substituted.

For free vibration problems with prestress, the vibration frequencies are

$$\omega=\{[\text{RHS}-\bar{N}_x(m\pi/a)^2-\bar{N}_y(n\pi/b)^2]/\rho\}^{1/2} \quad (17)$$

with mode shapes and fundamental natural frequency determined as for free vibrations. Numerical results for buckling and free vibration (without prestress) are presented and discussed in the next section.

Antisymmetric Laminate Results

In this section, some basic design analysis results for buckling and vibration of antisymmetrically laminated cross-ply rectangular plates will be presented and discussed. These results are essentially an extension of work done by Whitney.[1]

Buckling Results

The normalized buckling load for typical graphite/epoxy plates is plotted in Fig. 4 against the plate aspect ratio for 2, 4, 6, and an infinite number of layers. The parameter for the normalization, $\bar{N}_x b^2/(\pi^2 D_{22})$, can readily be isolated in Eq. (12) when the stiffnesses for antisymmetric cross-ply plates are substituted. The results for an infinite number of layers in Fig. 4 are the same as the orthotropic solution in which all coupling between bending and extension is ignored (then, B_{11}, the only coupling stiffness for an antisymmetric cross-ply, is zero). For a square two-layered plate, the actual results are about 65% below the orthotropic results. From the design analysis point of view, the orthotropic prediction is about 186% higher than the actual results. Note that as the number of layers increases from two, the buckling results rapidly approach the curve for the orthotropic solution. That is, coupling between bending and extension rapidly dies out as the number of layers increases. For example,

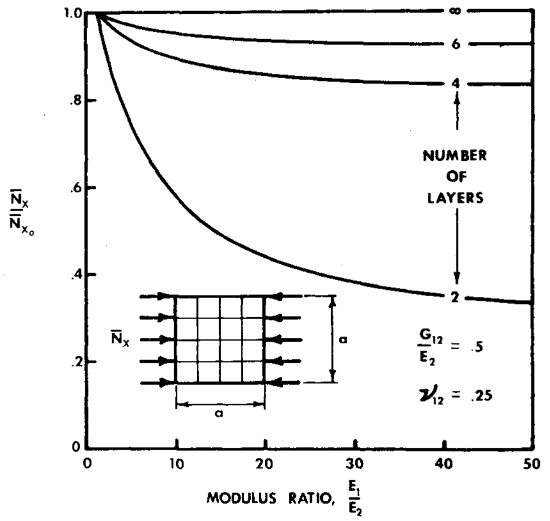

Fig. 5 Relative uniaxial buckling loads of square antisymmetric cross-ply laminated plates.

a four-layered plate (there must be an even number of layers to obtain antisymmetry) has a reduction of 16% and an over-prediction of 19% whereas a six-layered plate has a reduction of 7% and an 8% overprediction.

The cusps in Fig. 4 occur because of changes in the buckle mode shape as the plate aspect ratio changes. Note that the curves approach the lowest level of each respective curve at $a/b = 1$ as the aspect ratio increases. This behavior is typical of an isotropic plate as discussed by Timoshenko and Gere[7]; indeed, the present results reduce to their results when istropic plate stiffnesses are substituted in Eq. (12). The different aspect of the present results, the coupling between bending and extension, apparently has a constant influence irrespective of aspect ratio for a given number of layers. That is, the curves in Fig. 4 always have about the same relation to one another. For example, the minimum points on each curve for a specific number of layers have the same value; hence, each curve is always a fixed percentage higher or lower than any other curve.

When other composite materials are considered, the effect of coupling on the plate buckling load depends essentially on the material Young's modulus ratio, E_1/E_2, as shown in Fig. 5. There, the buckling load is normalized by the buckling load of an orthotropic plate ($B_{11} = 0$) for square plates. Values of G_{12}/E_2 and ν_{12} are fixed in Fig. 5 because their influence on the buckling load is small compared to that of E_1/E_2. As the modulus ratio decreases from the graphite/epoxy value of 40, the influence of coupling between bending and extension decreases slowly. As noted previously, the reduction in buckling load of a square two-layered graphite/epoxy plate from the orthotropic value is about 65%. For a square boron/epoxy plate the reduction is about 43%. From the design analysis point of view, the orthotropic solution is too high by 186% for a graphite/epoxy plate and by 74% for an analogous boron/epoxy plate. Obviously, coupling between bending and extension is extremely important when the plate has only two layers. However, the influence of coupling between bending and extension dies out rapidly as the number of layers increases. For example, the reduction in buckling load for a six-layered graphite/epoxy plate is only about 7% and about 5% for a boron/epoxy plate.

Vibration Results

The normalized fundamental (lowest) natural frequencies of rectangular antisymmetric cross-ply plates are plotted in Fig. 6 against the plate aspect ratio for a typical graphite/epoxy composite. The normalization parameter is readily identified in Eq. (12) when the stiffnesses are specialized to antisymmetric cross-ply laminates. Note that there are no mode shape changes such as occurred for buckling problems. The percentage differences between the results for various numbers of layers and the results for an orthotropic plate are about the same irrespective of the plate aspect ratio. The results for two-layered plates are about 60% of the orthotropic solution, about 91% for four-layered plates, and about 96% for six-layered plates.

Again, the effect of coupling between bending and extension rapidly dies out as the number of layers increases. The over-all influence of coupling between bending and extension is much the same as it is for buckling problems. The principal difference is that vibration results are less strongly affected because they involve the square root of essentially the right-hand side of Eq. (12) whereas buckling results involve the right-hand side itself. Thus, changes in the right-hand side due to coupling are effectively decreased for vibration problems as compared to buckling problems.

The effect of coupling between bending and extension on vibration for different materials is represented in Fig. 7 for square antisymmetric cross-ply plates as a function of modulus ratio. As for buckling problems, the largest effects occur for high modulus ratio materials. However, again, the effects for typical boron/epoxy materials ($E_1/E_2 = 10$) are nearly as significant as for graphite/epoxy materials ($E_1/E_2 = 40$).

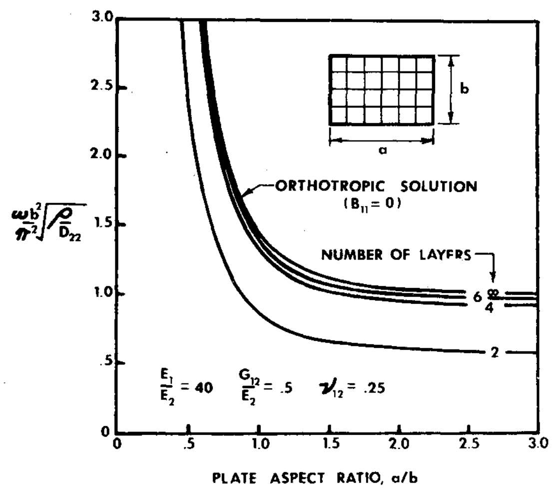

Fig. 6 Fundamental natural frequencies of rectangular antisymmetric cross-ply laminated plates.

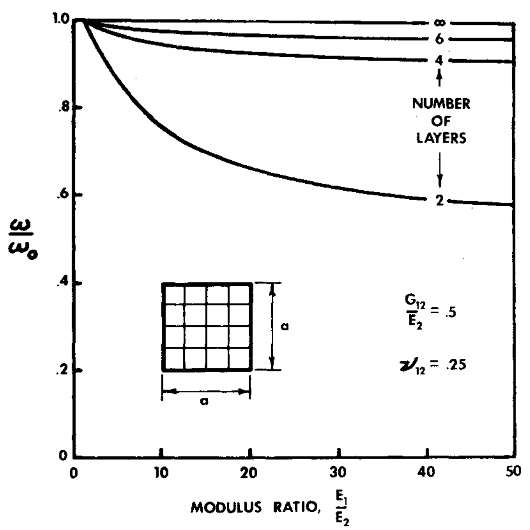

Fig. 7 Relative fundamental natural frequencies of square anti-symmetric cross-ply laminated plates.

Table 1 Buckling and vibration results for rectangular graphite/epoxy antisymmetric cross-ply laminated plates ($E_1/E_2 = 40$, $G_{12}/E_2 = 0.5$, $\nu_{12} = 0.25$)

a/b	Number of layers	$\bar{N}_x b^2/\pi^2 D_{22}$	$\omega b^2(\rho/D_{22})^{1/2}/\pi^2$
1	2	0.74781	0.86476
	4	1.7783	1.3335
	6	1.9691	1.4033
	∞ (orthotropic)	2.1218	1.4566
2	2	0.74781	0.60637
	4	1.7783	0.95479
	6	1.9691	1.0062
	∞ (orthotropic)	2.1218	1.0454
3	2	0.74781	0.58049
	4	1.7783	0.92393
	6	1.9691	0.97434
	∞ (orthotropic)	2.1218	1.0129

Specific numerical results for buckling and vibration corresponding to selected values in Figs. 4–7 are displayed in Tables 1 and 2 in order to provide a better basis for comparison with other numerical results than scaling curves. More significant figures than practically necessary are presented for ease of checking computer results (the results given were obtained on a Univac 1108).

Unsymmetric Laminate Results

Because of the infinite complexity of the class of laminates that are unsymmetric, general results are impossible. Instead, consider the cross sections of the contrived, but representative unsymmetric laminate example in Fig. 8. There, the fibers in the second layer from the bottom are oriented at 90° and the fibers in all other layers are oriented at 0° to the plate x-axis. Thus, for a constant thickness laminate, the 90° layer gets thinner and moves toward the bottom of the laminate as the number of layers increases. As mentioned previously, this example is contrived (i.e., probably not ever encountered in engineering practice), but it is a simple, straightforward example of unsymmetric laminates that is amenable to rather comprehensive theoretical parametric study.

Buckling Results

Normalized buckling loads for rectangular unsymmetric cross-ply laminated plates as a function of the number of layers are shown for typical graphite/epoxy composites in Fig. 9 and for boron/epoxy composites in Fig. 10. In both cases, the plate

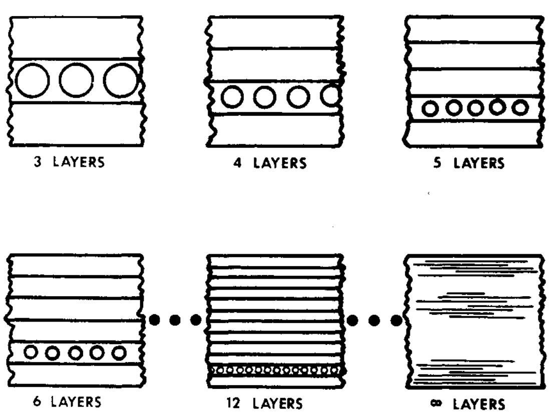

Fig. 8 Unsymmetric laminate example.

Table 2 Buckling and vibration results for square antisymmetric cross-ply laminated plates ($G_{12}/E_2 = 0.5$, $\nu_{12} = 0.25$)

E_1/E_2	Number of layers	$\bar{N}_x/\bar{N}_{x_o}$	ω/ω_o
10	2	0.57103	0.75567
	4	0.89276	0.94486
	6	0.95233	0.97588
	∞ (orthotropic)	1	1
40	2	0.35244	0.59367
	4	0.83811	0.91548
	6	0.92805	0.96335
	∞ (orthotropic)	1	1

aspect ratio is 2, a value for which the results are the most strikingly different from baseline results. One of the baseline comparison values is the buckling load for a laminate with all 0° layers. The other comparison case is an unsymmetric laminate for which coupling between bending and extension is ignored ($B_{ij} = 0$). Results for the actual unsymmetric laminate, for which coupling between bending and extension is considered, are labeled exact solution in the figures. As somewhat of a side issue, the flat spot in the $B_{ij} = 0$ curve in Fig. 10 is due to a buckling mode shape change.

More importantly, for both graphite/epoxy and boron/epoxy and for all numbers of layers, the actual laminate has, not surprisingly, less buckling resistance than a laminate with $B_{ij} = 0$. On the other hand, the actual laminate has more buckling resistance than a laminate with all 0° layers. The reason for this curious result is related to the relative values of the stiffnesses in the x- and y- directions and as well to the plate aspect ratio. As will be seen subsequently, D_{11} is decreased somewhat by the presence of a 90° layer, but D_{22} is increased by factors of up to an order of magnitude. The influence of D_{22} is most easily examined for a three-layered plate which is, of course, symmetric. Thus, the exact solution corresponds to the orthotropic solution ($B_{ij} = 0$), e.g., see the horizontal hash mark at three layers in Fig. 9 where the two curves coincide. At that point, the orthotropic solution for a plate with an aspect ratio (a/b) of two which buckles into the $m = 1$, $n = 1$ mode is

$$\bar{N}_x b^2 = \pi^2[D_{11}/4 + 2(D_{12} + 2D_{66}) + 4D_{22}] \tag{18}$$

which is readily obtained from Eq. (12). For a graphite/epoxy laminate, the term in Eq. (18) involving D_{11} decreases by less than 4% when the actual laminate is considered as opposed to the all 0° layer laminate. On the other hand, the term involving D_{22} is $2\frac{1}{2}$ times as big for the actual laminate as it is

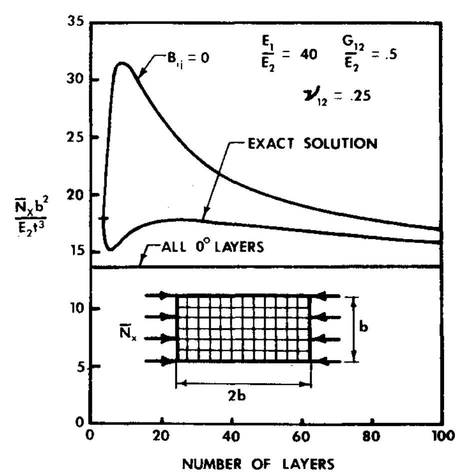

Fig. 9 Uniaxial buckling loads of rectangular unsymmetric cross-ply laminated plates (graphite/epoxy).

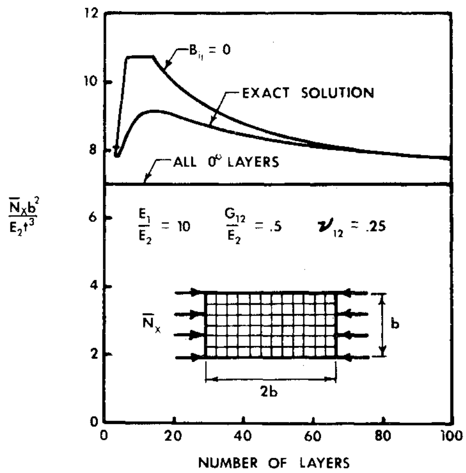

Fig. 10 Uniaxial buckling loads of rectangular unsymmetric cross-ply laminated plates (boron/epoxy).

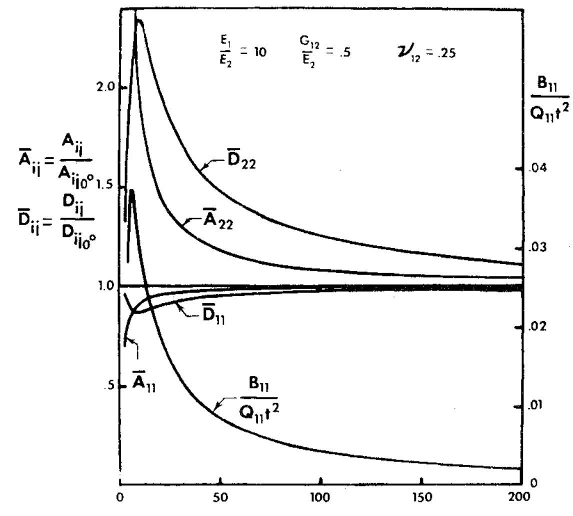

Fig. 12 Normalized stiffnesses of example unsymmetric cross-ply laminate (boron/epoxy).

for the all 0° layer laminate. The net result is 33% bigger for the actual laminate than for the all 0° layer laminate.

The differences between the exact, orthotropic, and all 0° layer predictions for graphite/epoxy laminates in Fig. 9 range from 48% less than the orthotropic ($B_{ij} = 0$) at 6 layers to 18% less at 40 layers to 6% less at 100 layers. In addition, the exact results range from about 30% more than the all 0° laminate results at 30 layers to about 18% more at 100 layers. Corresponding values for boron/epoxy laminates in Fig. 10 range from about 18% less than the orthotropic solution ($B_{ij} = 0$) at 6 layers to about 4% less at 40 layers to 1% less at 100 layers. Similarly, the exact results vary from about 31% more than the all 0° layer laminate results at 14 layers to about 10% more at 100 layers. Such differences are well within the consideration of usual engineering design practice. However, the differences exist for many more layers than coupling between bending and extension was believed to be important. That belief was established on the basis of extrapolating antisymmetric cross-ply results. Obviously, such extrapolation is invalid, but truly no one had anything better (except to remain skeptical or uncertain and try to find out what actually happens for *unsymmetrical* laminates).

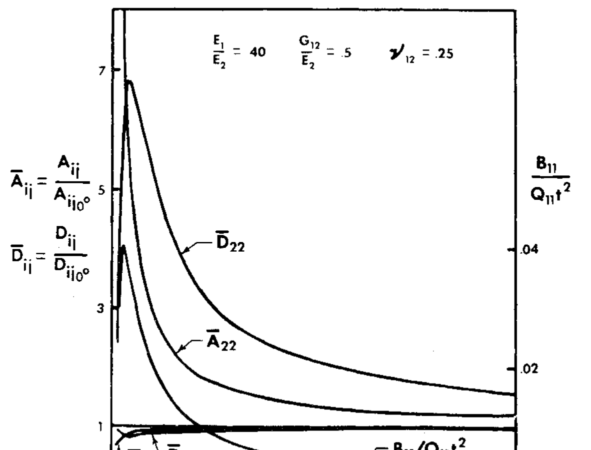

Fig. 11 Normalized stiffnesses of example unsymmetric cross-ply laminate (graphite/epoxy).

As an aid to understanding the buckling behavior illustrated in Figs. 9 and 10, the extensional, coupling, and bending stiffnesses are plotted vs the number of layers in Figs. 11 and 12 for graphite/epoxy and boron/epoxy, respectively. The stiffnesses in the x-direction (with which most fibers are aligned), $\bar{A}_{11}$ and $\bar{D}_{11}$, are nearly independent of the number of layers. However, the 90° layer causes the stiffnesses in the y-direction, $\bar{A}_{22}$ and $\bar{D}_{22}$, to deviate by up to an order of magnitude from the value for all 0° layers (which is the same as the value for an infinite number of layers). These discrepancies die out very slowly as the number of layers increases. Moreover, the normalized stiffness for coupling between bending and extension, which can be shown to be

$$B_{11}/(Q_{11}t^2) = \{1/[2(NL)^2]\}(1 - E_2/E_1)(NL - 3) \quad (19)$$

(where NL is the number of layers) and which appears in Eq. (12) and thereby enables stiffnesses A_{22} and D_{22} to influence the buckling load, also dies out slowly. The maximum coupling occurs for this unsymmetric laminate when $NL = 6$. Do not attempt to compare the magnitudes of, e.g.,$\bar{D}_{22}$ and $B_{11}/(Q_{11}t^2)$ quantitatively. They do not have the same base for normalization (B_{11} cannot be normalized relative to its value for all 0° layers since there its value is zero). Thus, it is easy to see why the effect of a single unsymmetrically placed 90° layer on the buckling load dies out slowly as the number of layers increases. This conclusion is particularly striking for graphite/epoxy, but is no less valid for boron/epoxy.

For unsymmetric cross-ply laminates wherein each layer is of the same material but of different orientation (either 0° or 90°, but not necessarily alternating), B_{12} and B_{66} can easily be shown to be zero. Also, B_{11} and B_{22} are not of equal value but opposite sign as for antisymmetric cross-ply laminates. However, for general unsymmetric laminates of different materials, all coupling stiffnesses can occur. The present theory is applicable to the general lamination case as long as the principal material directions of each layer are aligned with the plate sides.

Vibration Results

Normalized fundamental natural frequencies for the example unsymmetrical cross-ply laminated graphite/epoxy and boron/epoxy plates are shown in Figs. 13 and 14, respectively. The vibration results are analogous to the buckling results in the same manner as explained for antisymmetric laminates (lesser differences for vibration results because of a square root factor). The differences between the present exact solution and the alternative design analysis approaches of using an orthotropic

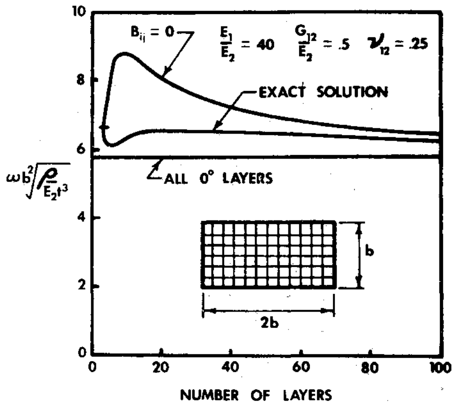

Fig. 13 Fundamental natural frequencies of rectangular unsymmetric cross-ply laminated plates (graphite/epoxy).

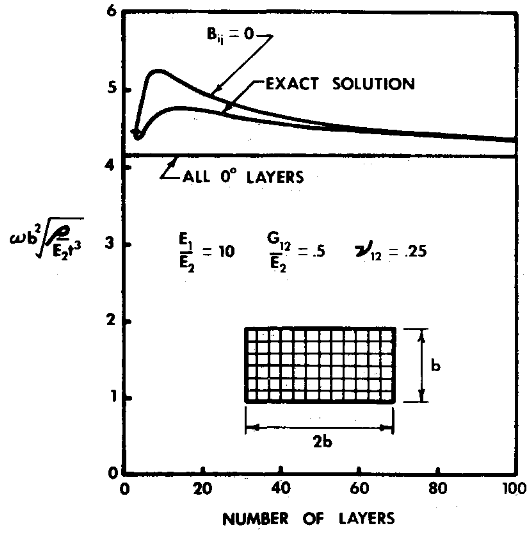

Fig. 14 Fundamental natural frequencies of rectangular unsymmetric cross-ply laminated plates (boron/epoxy).

solution ($B_{ij} = 0$) or all 0° layers are clearly significant enough to warrant consideration in engineering practice.

Concluding Remarks

A new exact theory for buckling and vibrations of unsymmetrically laminated cross-ply rectangular plates is presented and discussed. The plates can be subjected to arbitrary combinations of biaxial loading either to cause buckling or to incur a state of initial stress during vibration. Classical laminated plate theory including the Kirchhoff hypothesis of nondeformable normals is used for *S*2 type simply supported edge boundary conditions. Numerical examples for common boron/epoxy and graphite/epoxy composite materials are used to display results of the theory. Both antisymmetric laminates and the more general case of unsymmetric laminates are addressed in the numerical examples.

For antisymmetric laminates, new buckling results are presented in which coupling between bending and extension causes buckling loads that are lower than orthotropic predictions by 65% for square graphite/epoxy plates. Conversely, the orthotropic predictions are too high by about 186%. Somewhat smaller differences occur for boron/epoxy plates. Analogous vibration results with smaller differences than the buckling results are also presented as a minor correction of some of Whitney's numerical results. For both buckling and vibration of antisymmetric cross-ply plates, the effect of coupling between bending and extension dies out rapidly as the number of layers increases to the point where for many practical laminates it is unimportant.

For unsymmetric laminates, a contrived example was used to illustrate what can happen if coupling between bending and extension is ignored. The special example is necessary because general results cannot be presented for such a class of laminates with infinite variety. For both buckling and vibration problems, the effect of coupling between bending and extension decreases very *slowly* with increasing number of layers in contrast to the rapid decrease for antisymmetric laminates. The effect of coupling is about 20% for buckling and about 10% for vibration of the example laminate with *forty* layers. Thus, designers will have to account for coupling between bending and extension in *all* practical unsymmetric laminates.

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